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A Geometric Representation of the Morse Fan

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Abstract. It was proved in Jongen and Pallaschke (1988) that every piecewise smooth Morse function f defined on an open subset of $\mathbb{R}^n$ can be represented in suitable coordinates in the neighborhood of a nondegenerate critical point as $f(x_0)+l(y_1,\dots,y_k)-\sum_{i=k+1}^{k+\mu} y_i^2+\sum_{j=k+\mu+1}^{n} y_j^2$, where the piecewise linear function $l\in CS(y_1,\dots,y_k,-\sum_{i=1}^{k} y_i)$ is a continuous selection of the coordinate functions $y_1,\dots,y_k$ and their negative sum $-\sum_{i=1}^{k} y_i$. In this paper we study a collection of cones in $\mathbb{R}^k$ on which the functions $l\in CS(y_1,\dots,y_k,-\sum_{i=1}^{k} y_i)$ are linear. This collection of cones forms a complete polyhedral fan and will be called the Morse fan. It is shown that Morse fan is a refinement of the normal fan of the polytope $\mathbb{C}P$ which is the Minkowski sum of two pyramids $\mathbb{P}$ and $-\mathbb{P}$, where $\mathbb{P}=\mathrm{conv}\{e_1,\dots,e_k,-\sum_{i=1}^{k} e_i\}$ is the convex hull of the unit vectors $e_1,\dots,e_k\in\mathbb{R}^k$ and their negative sum.

Key words: Combinatorial convexity, Nonsmooth Morse theory, Nonsmooth optimization

1. Introduction

Let $U\subseteq\mathbb{R}^n$ be an open subset and $f,f_1,\dots,f_m: M\longrightarrow\mathbb{R}$ be continuous functions. If $I(x)=\{i\in\{1,\dots,m\}\,|\,f_i(x)=f(x)\}$ is nonempty at every point $x\in U$, then f is called a *continuous selection* of the functions $f_1,\dots,f_m$. We denote by $CS(f_1,\dots,f_m)$ the set of all continuous selections of $f_1,\dots,f_m$. The set $I(x)$ is called the active index set of f at the point x. Typical examples for continuous selections are the functions

$$f_{\max}=\max(f_1,\dots,f_m),\quad f_{\min}=\min(f_1,\dots,f_m)$$

or more generally any function obtained from $f_1,\dots,f_m$ by exploiting finitely many times the operation of taking maximum or minimum.

The notion of a nondegenerate critical point for a continuous selections of C^2-functions has been defined in Jongen and Pallaschke (1988) and the following generalization of the second Morse Lemma for a continuous selection of C^2-functions was proved:

THEOREM 1.1 *Let* $U \subseteq \mathbb{R}^n$ *be an open subset,* $f_1, \ldots, f_m : U \longrightarrow \mathbb{R}$ *be twice continuously differentiable functions, and let* $x_0 \in U$ *be a nondegenerate critical point of* $f \in CS(f_1, \ldots, f_m)$. *Then* f *is locally topologically equivalent in a neighborhood of* x_0 *to a function of the form*

$$f(x_0) + l(y_1, \ldots, y_k) - \sum_{i=k+1}^{k+\mu} y_i^2 + \sum_{j=k+\mu+1}^{n} y_j^2,$$

with $k = |\hat{I}(x_0)| - 1$, *where* $\hat{I}(x_0) = \{j \in I(x_0) \mid x \in \mathrm{cl}(\mathrm{int}(\{z \mid f(z) = f_j(z)\}))\}$ *is the essential active index set,* $l \in CS(y_1, \ldots, y_k, -\sum_{i=1}^{k} y_i)$, *and* μ *the quadratic index of* f *at* x_0.

For more details see Jongen et al. (2000), Jongen and Pallaschke (1988) and Agrachev et al. (1997). The following theorem was proved by Bartels et al. (1995) (see also Melzer, 1986):

THEOREM 1.2 *Let* $l \in CS(l_1, \ldots, l_{m+1})$ *be a continuous selection of the functions* $l_i(y) = y_i$ *for* $i \in \{1, \ldots, m\}$ *and* $l_{m+1}(y) = -\sum_{i=1}^{m} y_i$ *with* $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$. *Then the following statements hold*:

(i) *l has a unique max-min representation*

$$l(x) = \max_{i \in \{1, \ldots, r\}} \min_{j \in M_i} l_j(x),$$

where the index sets $M_1, \ldots, M_r$ *with* $M_i \subseteq \{1, \ldots, m+1\}$ *are such that* $M_i \subseteq M_j$ *if and only if* $i = j$.

(ii) *l is representable as the difference of two sublinear functions*:

$$l(x) = \max_{i \in \{1, \ldots, r\}} \min_{j \in M_i} l_j(x) = \max_{i \in \{1, \ldots, r\}} \{\sum_{\substack{k=1 \\ k \neq i}}^{r} \max_{j \in M_k} -l_j(x)\} - \sum_{k=1}^{r} \max_{j \in M_k} -l_j(x).$$

For applications to nonsmooth optimization we refer to Demyanov and Rubinov (1986), Jongen, Jonker et al. (2000), Pallaschke and Rolewicz (1997) and Pallaschke and Urbański (2000).

2. The Morse Fan

For a nonempty set $Z \subset \mathbb{R}^n$ the set of all nonnegative linear combinations

$$\sigma = \left\{ \sum_{i=1}^{r} a_i z_i \mid a_i \in \mathbb{R} \text{ and } a_i \geqslant 0,\ z_i \in Z, i \in \{1, \ldots, r\}, r \in \mathbb{N} \right\} \subset \mathbb{R}^n$$

is called the cone determined by Z. If the set $Z = \{z_1, \ldots, z_r\}$ is finite then σ is called a *polyhedral cone* determined by $z_1, \ldots, z_r \in \mathbb{R}^n$. For a cone $\sigma \subset \mathbb{R}^n$ we call a cone $\tau \subset \sigma$ a *face* of σ if for every $x, y \in \sigma$ and some $t \in (0,1)$ the condition $tx+(1-t)y \in \tau$ implies that $x, y \in \tau$. Note that for every cone σ the apex $\{0\}$ and the cone σ itself are faces of σ.

A *fan* in $\mathbb{R}^n$ is a finite collection

$$\Sigma = \{\sigma_1, \ldots, \sigma_s\}$$

of nonempty cones with the following properties:

(i) Every face of $\sigma \in \Sigma$ is again an element of Σ.

(ii) The intersection $\sigma \cap \sigma'$ of any two cones σ, $\sigma' \in \Sigma$ is a face of both σ and σ'.

A fan $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ in $\mathbb{R}^n$ is called *polyhedral* if each of its cones is a polyhedral cone, *simplicial* if each of its cones is the nonnegative linear combination of linearly independent vectors and *complete* if its cones cover $\mathbb{R}^n$, i.e., $\bigcup_{i=1}^{s} \sigma_i = \mathbb{R}^n$. For more details see Ewald (1996).

In Bartels et al. (1995) a collection of cones in $\mathbb{R}^n$ on which every $l \in CS(y_1, \ldots, y_n, -\sum_{i=1}^n y_i)$ is linear has been studied. This cones are constructed in the following way: Put $l_i(x) = x_i$ for $i \in \{1, \ldots, n\}$ and $l_{n+1}(x) = -\sum_{i=1}^n x_i$, with $x \in \mathbb{R}^n$ and denote by Π_{n+1} the set of all permutations of the numbers $1, \ldots, n+1$. For a permutation $\pi \in \Pi_{n+1}$ the set

$$\sigma_\pi = \{x \in \mathbb{R}^n \mid l_{\pi(1)}(x) \leqslant l_{\pi(2)}(x) \leqslant \cdots \leqslant l_{\pi(n+1)}(x)\}$$

is a cone, called *permutation cone*. It has been shown in Bartels et al. (1995), that all cones σ_π have nonempty interiors. Furthermore note that $\bigcup_{\pi \in \Pi_{n+1}} \sigma_\pi = \mathbb{R}^n$.

Now we define the *Morse fan*

$$\Sigma_n = \{\tau \subset \mathbb{R}^n \mid \tau \text{ is a face of } \sigma_\pi,\ \pi \in \Pi_{n+1}\}$$

as the collection of all faces of the above defined permutation cones σ_π.

It follows immediately from the definition that Σ_n is a complete fan in $\mathbb{R}^n$.

Minimal representations for the elements of $CS(y_1, y_2, y_3, -\sum_{i=1}^3 y_i)$ as differences of sublinear functions are given in Grzybowski, Pallaschke and Urbański (2000) and the combinatorial Picard group of Σ_n has been studied in Pallaschke and Rolewicz (1999).

PROPOSITION 2.1 *For every* $n \in \mathbb{N}$ *the fan* Σ_n *has* $(2^{n+1}-2)$ *different one-dimensional cones which are generated by the following vectors*:

– $\mathbf{1} = (1, \ldots, 1) = \sum_{i=1}^{n} e_i$

– $x_M = -m\mathbf{1} + (n+1)\sum_{i \in M} e_i$ *for* $M \subseteq \{1, .., n\}$ *and* $m = \text{card } M \geqslant 1$

and its negatives, where " card*" denotes the cardinality of a set.*

Proof. The one-dimensional cones of Σ_n are contained in the solution spaces of all subsystems of $(n-1)$ equations of the from

$$x_i = x_j \quad \text{for} \quad i, j \in \{1, \ldots, n\}, \quad i < j$$

and

$$x_i = -\sum_{j=1}^{n} x_j \quad \text{for} \quad i \in \{1, \ldots, n\},$$

which have full rank. If such a system of $(n-1)$ linear equations is written in matrix notation as $Ax = 0$, then we have two types of row vectors in the matrix A:

The row vector, which corresponds to the equation $x_i = x_j$ for $i<j$, is of the type:

a) $(0, 0, \ldots, 0, -1, 0, \ldots, 1, 0, 0, 0)$
and the row vector, which corresponds to the equation $x_i = -\sum_{j=1}^{n} x_j$, is of the type:
b) $(-1, -1, \ldots, -1, -2, -1, \ldots, -1, -1)$.

Since the difference of two row-vectors of type b) is a row-vector of type a), it follows that an $(n-1, n)$-matrix A of arbitrary row-vectors of type a) and b) has full rank if and only if no diagonal element is equal to 0. Hence, up to permutations of variables and rows, we have to consider the following linear equations $Ax = 0$.

Assume that the matrix A consists only of vectors of type b):

$$A = \begin{pmatrix} -2 & -1 & -1 & -1 & -1 & . & . & . & . & . & -1 \\ -1 & -2 & -1 & -1 & -1 & . & . & . & . & . & -1 \\ -1 & -1 & -2 & -1 & -1 & . & . & . & . & . & -1 \\ . & . & . & . & . & . & . & . & . & . & . \\ -1 & -1 & . & . & -2 & -1 & . & . & . & . & -1 \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & -1 \end{pmatrix}.$$

Then the solution space of $Ax = 0$ is

$$\lambda(-1, -1, -1, \ldots, -1, n), \qquad \lambda \in \mathbb{R},$$

and by permuting the variables $x_1, \ldots, x_n$ we get all n solutions.

Assume that the matrix A consists of vectors of type b) and of exactly one vector of type a). Since the difference of two vectors of type b) is a vector of type a), we get up to permutation the matrix

$$A=\begin{pmatrix} -2 & -1 & -1 & -1 & -1 & . & . & . & . & . & -1 \\ -1 & -2 & -1 & -1 & -1 & . & . & . & . & . & -1 \\ -1 & -1 & -2 & -1 & -1 & . & . & . & . & . & -1 \\ . & . & . & . & . & . & . & . & . & . & . \\ -1 & -1 & . & . & -2 & -1 & . & . & . & . & -1 \\ . & . & . & . & . & . & . & . & . & . & . \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

In this case the solution space of $Ax=0$ is

$$\lambda(-2,-2,-2,\dots,-2,n-1,n-1), \qquad \lambda\in\mathbb{R},$$

and by permuting the variables $x_1,\dots,x_n$ we get all $\binom{n}{2}$ solutions.

If we continue in this way then we get, up to permutations, the solution spaces:

$$\begin{aligned} &\lambda(-3,-3,-3,\dots,-3,n-2,n-2,n-2,n-2)\ , && \lambda\in\mathbb{R} \\ &\lambda(-4,-4,-4,\dots,-4,n-3,n-3,n-3,n-3)\ , && \lambda\in\mathbb{R} \\ &\qquad . \\ &\qquad . \\ &\lambda(-n+1,2,2,\dots,2,2,2,2,2) && ,\ \lambda\in\mathbb{R}. \end{aligned}$$

If the matrix consists only of row-vectors of type a) and has full rank, then there exists a permutation such that all elements in the diagonal are -1, hence

$$A=\begin{pmatrix} -1 & 0 & 0 & 0 & . & . & 1 & . & . & . & . & 0 \\ 0 & -1 & -1 & 0 & 0 & . & . & . & 1 & . & . & 0 \\ 0 & 0 & 0 & -1 & 1 & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & . & -1 & 1 \end{pmatrix}$$

and the solution space is

$$\lambda(1,1,1,\dots,1), \qquad \lambda\in\mathbb{R},$$

which proves the proposition. □

REMARK 2.2 It follows from Proposition 2.1 that the Morse fan Σ_n is a polyhedral fan, because Σ_n has only finitely many one-dimensional cones and every cone of Σ_n is the nonnegative linear combination of vectors which generate the one-dimensional cones.

3. The Configuration Polytope

Let $M \subset \{1, \ldots, n\}$ be a set with cardinality m with $1 \leqslant m \leqslant n$. Then we define

$$a_M = (a_1, \ldots, a_n) = -m\mathbf{1} + (n+1)\sum_{j \in M} e_j$$

where $e_i \in \mathbb{R}^n$ is the i-th unit vector and $\mathbf{1} = \sum_{i=1}^{n} e_i$. Observe that $a_i = -m$ for the components $i \in \{1, \ldots, n\} \setminus M$ and that $a_i = n+1-m$ for $i \in M$.

Let us put

$$\mathcal{J} = \{a \in \mathbb{R}^n \mid \text{there exits} \quad M \subset \{1, \ldots, n\} \quad \text{with} \quad a = a_M \quad \text{or} \quad a = -a_M\}.$$

Now we put

$$\mathbb{P} = \operatorname{conv}\{e_1, \ldots, e_n, -\sum_{i=1}^{n} e_i\} \subset \mathbb{R}^n$$

and call

$$\mathbb{C}\mathrm{P} = \{x \in \mathbb{R}^n \mid \langle a, x \rangle \leqslant n+1 \quad \text{with} \quad a \in \mathcal{J}\ \}$$

the *configuration polytope*, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$.

Now the following statement holds:

PROPOSITION 3.1 *Let X be a Hausdorff topological vector space and let A, B be closed convex subsets of X such that $0 \in \operatorname{int} A$ and $A \subset B$. If the boundary ∂B contains ∂A then $A = B$.*

Proof. Let us assume that $x \in B \setminus A$. Then there exists the greatest $\lambda \in (0, 1)$ such that $\lambda x \in A$. Then $\lambda x \in \partial A \subset \partial B$.

On the other hand, there exists a neighborhood U of 0 which is contained in B. Then $\lambda x + (1-\lambda)U \subset B$ is a neighborhood of λx and $\lambda x \in \operatorname{int} B$. Hence $x \notin B$ which contradicts our assumption. □

REMARK 3.2 Let

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}, f(x) = \max\{\langle a, x \rangle, a \in \mathcal{J}\}.$$

Then

$$\mathbb{C}\mathrm{P} = \{x \mid f(x) \leqslant n+1\} \text{ and } \partial\mathbb{C}\mathrm{P} = \{x \mid f(x) = n+1\}.$$

This remark is trivial. The second equality follows from the fact that $f(0) = 0$.

THEOREM 3.3 *For the configuration polytope holds*:

$$\mathbb{C}\mathrm{P} = \mathbb{P} - \mathbb{P}.$$

Proof. Let us first prove that $\mathbb{P}-\mathbb{P}\subseteq\mathbb{CP}$ holds. Therefore let us notice that $\mathbb{P}-\mathbb{P}=\mathrm{conv}\big(\{e_i-e_j\mid i,j\in\{1,\dots,n\},i\neq j\}\cup\{1+e_i\mid i=1,\dots,n\}\cup\{-1-e_i\mid i=1,\dots,n\}\big)$.

Now let $M\subset\{1,\dots,n\}$ with $\mathrm{card}(M)=m$. If $i\in M$ then $\langle a_M,e_i\rangle=n+1-m$. If $i\notin M$ then $\langle a_M,e_i\rangle=-m$. Also $\langle a_M,-\mathbf{1}\rangle=-m$. Therefore, $\langle a_M,e_i-e_j\rangle\in\{0,n+1,-n-1\}$ for all $M\subset\{1,\dots,n\}$.

Moreover, $\langle a_M,\mathbf{1}+e_i\rangle\in\{0,n+1\}$ and $\langle a_M,-\mathbf{1}-e_i\rangle\in\{0,-n-1\}$.

Hence for all $a\in\mathcal{J}$ and all vertices b of $\mathbb{P}-\mathbb{P}$, $\langle a,b\rangle\in\{0,n+1,-n-1\}$. Therefore $\mathbb{P}-\mathbb{P}\subset\mathbb{CP}$.

Now we prove the reverse inclusion: Let A,B be faces of $\mathbb{P}$. Let $A=\mathrm{conv}\{a_1,\dots,a_p\}$ and $B=\mathrm{conv}\{b_1,\dots,b_q\}$ where $a_i,b_i\in\{e_1,\dots,e_n,-\mathbf{1}\}$. If $a_i=b_j$ for some i,j then $0\in A-B$. Since $0\in\mathrm{int}(\mathbb{P}-\mathbb{P})$ then $A-B$ is not a face of $\mathbb{P}-\mathbb{P}$.

Let us assume that

$$\{a_1,\dots,a_p\}\cap\{b_1,\dots,b_q\}=\emptyset,\ \{a_1,\dots,a_p\}\cup\{b_1,\dots,b_q\}=\{e_1,\dots,e_n,-\mathbf{1}\},\\ \text{and}\, b_q=-\mathbf{1}.$$

Let us denote, only for this part of the proof, the set $\{i\in\{1,\dots,n\}\mid\ e_i\in\{a_1,\dots,a_p\}\}$ by $\mathcal{J}$. Then $\langle a_J,a_i\rangle=n+1-p, i=1,\dots,p$ and $\langle a_J,b_i\rangle=-p,\quad i\in\{1,\dots,q\}$. Since $A-B=\mathrm{conv}\{a_i-b_j\mid i\in\{1,\dots,p\};\quad j=1,\dots,q\}$ and $\langle a_J,a_i-b_j\rangle=n+1$ then $\langle a_J,x\rangle=n+1, x\in A-B$ and $f(x)\geqslant n+1, x\in A-B$. Each face C of $\mathbb{P}-\mathbb{P}$ is a Minkowski sum of faces of $\mathbb{P}$ and $-\mathbb{P}$. Then C is contained in some $A-B$ or $B-A$ which was described above. Hence $f(x)\geqslant n+1, x\in C$. But according to Proposition 3.1 and Remark 3.3 $f(x)=n+1, x\in C$. The boundary $\partial(\mathbb{P}-\mathbb{P})$ is the union of all faces of $\mathbb{P}-\mathbb{P}$ which implies that $\partial(\mathbb{P}-\mathbb{P})\subset\partial\mathbb{CP}$ and, according to Proposition 3.2, $\mathbb{CP}=\mathbb{P}-\mathbb{P}$. □

Next we prove several elimination rules for the constraints of $\mathbb{CP}$.

PROPOSITION 3.4 *Let* $\mathrm{x}_0=(x_1^0,\dots,x_n^0)\in\mathbb{CP}=\{x\in\mathbb{R}^n\mid\langle a,x\rangle\leqslant n+1$ *with* $a\in\mathcal{J}$ $\}$ *be a feasible point of the configuration polytope. Then the following properties hold:*

(i) *If for two sets* $K, M \subset \{1, \dots, n\}$ *with* $1 \leqslant k = \mathrm{card}(K), m = \mathrm{card}(M) \leqslant (n-1)$ *the relations*

$$
\begin{aligned}
-k\sum_{i=1}^{n} x_i^0 \quad &+ \quad (n+1)\sum_{i\in K} x_i^0 = n+1,\\
-m\sum_{i=1}^{n} x_i^0 \quad &+ \quad (n+1)\sum_{i\in M} x_i^0 = n+1
\end{aligned}
$$

hold, then $K \cap M \neq \emptyset$ *and*

$$
\begin{aligned}
-r\sum_{i=1}^{n} x_i^0 \quad &+ \quad (n+1)\sum_{i\in K\cap M} x_i^0 = n+1,\\
-(k+m-r)\sum_{i=1}^{n} x_i^0 \quad &+ \quad (n+1)\sum_{i\in K\cup M} x_i^0 = n+1
\end{aligned}
$$

with $r = \mathrm{card}(K \cap M)$.

(ii) *If for two sets* $K, M \subset \{1, \dots, n\}$ *with* $1 \leqslant k = \mathrm{card}(K), m = \mathrm{card}(M) \leqslant (n-1)$ *and* $K \cap M \neq \emptyset$ *the relations*

$$
\begin{aligned}
-k\sum_{i=1}^{n} x_i^0 \quad &+ \quad (n+1)\sum_{i\in K} x_i^0 = n+1,\\
m\sum_{i=1}^{n} x_i^0 \quad &- \quad (n+1)\sum_{i\in M} x_i^0 = n+1
\end{aligned}
$$

hold, then

$$
\begin{aligned}
-(k-r)\sum_{i=1}^{n} x_i^0 \quad &+ \quad (n+1)\sum_{i\in K\setminus M} x_i^0 = n+1,\\
(m-r)\sum_{i=1}^{n} x_i^0 \quad &- \quad (n+1)\sum_{i\in M\setminus K} x_i^0 = n+1
\end{aligned}
$$

with $r = \mathrm{card}(K \cap M)$.

(iii) *If for an index* $i^* \in \{1, \dots, n\}$ *the constraint*

$$
\sum_{i=1}^{n} x_i^0 \quad - \quad (n+1)x_{i^*}^0 \quad = \quad n+1
$$

is satisfied, then for all subsets $M \subset \{1, \ldots, n\}$ *with* $2 \leqslant m = \mathrm{card}(M) \leqslant (n-1)$ *and* $i^* \in\ m$ *the strict inequality*

$$-m\sum_{i=1}^{n} x_i^0 \quad + \quad (n+1)\sum_{i \in M} x_i^0 \quad < \quad n+1$$

holds.

(iv) *If the constraint* $\sum_{i=1}^{n} x_i^0 \ = \ n+1$ *is satisfied, then for all subsets* $M \subset \{1, \ldots, n\}$ *with* $2 \leqslant m = \mathrm{card}(M) \leqslant (n-1)$ *the strict inequality*

$$m\sum_{i=1}^{n} x_i^0 \quad - \quad (n+1)\sum_{i \in M} x_i^0 \quad < \quad n+1$$

holds.

REMARK 3.5 Observe that conditions similar to (i) and (iii)-vi) hold if the constraints of the type

$$-k\sum_{i=1}^{n} x_i^0 \quad + \quad (n+1)\sum_{i \in K} x_i^0 \quad = \quad n+1$$

are replaced by constraints of the type

$$k\sum_{i=1}^{n} x_i^0 \quad - \quad (n+1)\sum_{i \in K} x_i^0 \quad = \quad n+1$$

and $\sum_{i=1}^{n} x_i^0 \ = \ n+1$ by the constraint $-\sum_{i=1}^{n} x_i^0 \ = \ n+1$.

Proof. Let us assume that $x_0 \in \mathbb{C}\mathrm{P}$ is a feasible point and that:

(i) for two sets $K, M \subset \{1, \ldots, n\}$ with $1 \leqslant k = \mathrm{card}(K), m = \mathrm{card}(M) \leqslant (n-1)$ the relations

$$-k\sum_{i=1}^{n} x_i^0 \quad + \quad (n+1)\sum_{i \in K} x_i^0 = n+1,$$

$$-m\sum_{i=1}^{n} x_i^0 \quad + \quad (n+1)\sum_{i \in M} x_i^0 = n+1$$

hold. Adding both equations gives:

$$-(k+m)\sum_{i=1}^{n} x_i^0 + 2(n+1)\sum_{s \in K \cap M} x_s^0 + (n+1)\sum_{m \in\ M \setminus K} x_m^0 + (n+1)\sum_{k \in K \setminus M} x_k^0 = 2(n+1).$$

Now assume that $K \cap M = \emptyset$. Then we get

$$-(k+m)\sum_{i=1}^{n} x_i^0 + (n+1)\sum_{m\in\ M\cup K} x_m^0 = 2(n+1)$$

and this is not possible for a feasible point. Hence $K \cap\ M \neq \emptyset$.
If we put $r = \mathrm{card}(K \cap\ M)$ then:

$$\begin{aligned}
2(n+1)\ &= -(k+m)\sum_{i=1}^{n} x_i^0 + 2(n+1)\sum_{s\in K\cap M} x_s^0 \\
&\quad + (n+1)\sum_{m\in\ M\setminus K} x_m^0 + (n+1)\sum_{k\in K\setminus M} x_k^0 \\
&= -r\sum_{i=1}^{n} x_i^0 \quad + \quad (n+1)\sum_{i\in K\cap M} x_i^0 \\
&\quad -(k+m-r)\sum_{i=1}^{n} x_i^0 \quad + \quad (n+1)\sum_{i\in K\cup M} x_i^0
\end{aligned}$$

Since both summands are active hyperplanes in $x_0 \in \mathbb{C}\mathrm{P}$ we get:

$$\begin{aligned}
-r\sum_{i=1}^{n} x_i^0 \quad + \quad (n+1)\sum_{i\in K\cap M} x_i^0 &= n+1 \\
-(k+m-r)\sum_{i=1}^{n} x_i^0 \quad + \quad (n+1)\sum_{i\in K\cup M} x_i^0 &= n+1
\end{aligned}$$

(ii) for two sets $K, M \subset \{1, \ldots, n\}$ with $1 \leqslant k = \mathrm{card}(K), m = \mathrm{card}(M) \leqslant (n-1)$ and $K \cap\ M \neq \emptyset$ the relations

$$\begin{aligned}
-k\sum_{i=1}^{n} x_i^0 \quad + \quad (n+1)\sum_{i\in K} x_i^0 &= n+1, \\
m\sum_{i=1}^{n} x_i^0 \quad - \quad (n+1)\sum_{i\in M} x_i^0 &= n+1
\end{aligned}$$

hold. Since the relations

$$-k\sum_{i=1}^{n}x_i^0+(n+1)\sum_{i\in K}x_i^0= \quad -k\sum_{i\in\{1,..,n\}\setminus K}x_i^0$$
$$+\ (n+1-k)\sum_{i\in K}x_i^0,$$
$$m\sum_{i=1}^{n}x_i^0-(n+1)\sum_{i\in M}x_i^0=-(n+1-m)\sum_{i\in M}x_i^0$$
$$+\quad m\sum_{i\in\{1,..,n\}\setminus M}x_i^0$$

hold, we have for $r=\mathrm{card}(K\cap M)$:

$$(m-k)\sum_{i=1}^{n}x_i^0+(n+1)\left(\sum_{i\in K}x_i^0-\sum_{i\in M}x_i^0\right)$$
$$=\ (m-k)\sum_{i\in\{1,\dots,n\}\setminus(K\cup M)}x_i^0+[(n+1)+(m-k)]\sum_{i\in(K\setminus M)}x_i^0$$
$$+[-(n+1)+(m-k)]\sum_{i\in(M\setminus K)}x_i^0+r\sum_{i\in(K\cap m)}x_i^0$$
$$=\ -(k-r)\sum_{i=1}^{n}x_i^0+(n+1)\sum_{i\in K\setminus M}x_i^0$$
$$+(m-r)\sum_{i=1}^{n}x_i^0-(n+1)\sum_{i\in M\setminus K}x_i^0.$$

Since the last two summands are active hyperplanes in $x_0\in\mathbb{C}\mathrm{P}$ it follows that

$$-(k-r)\sum_{i=1}^{n}x_i^0+(n+1)\sum_{i\in K\setminus M}x_i^0=n+1,$$
$$(m-r)\sum_{i=1}^{n}x_i^0-(n+1)\sum_{i\in M\setminus K}x_i^0=n+1$$

holds with $r=\mathrm{card}(K\cap M)$.

iv) the constraint $\sum_{i=1}^{n}x_i^0=n+1$ is satisfied.

Let us furthermore assume that for a subset $M\subset\{1,\dots,n\}$ with $2\leqslant m=\operatorname{card}(M)\leqslant(n-1)$ the equation

$$m\sum_{i=1}^{n}x_i^0-(n+1)\sum_{i\in M}x_i^0=n+1$$

holds. Then the sum of the two equations

$$\begin{aligned}\sum_{i=1}^{n}x_i^0 \qquad\qquad &= n+1,\\ m\sum_{i=1}^{n}x_i^0 \quad - \quad (n+1)\sum_{i\in M}x_i^0 &= n+1\end{aligned}$$

gives:

$$(m+1)\sum_{i\in\{1,\dots,n\}\setminus M}x_i^0 \quad - \quad (n-m)\sum_{i\in M}x_i^0=2(n+1)$$

which is a contradiction, since

$$\begin{aligned}&(m+1)\sum_{i\in\{1,\dots,n\}\setminus M}x_i^0 \quad - \quad (n-m)\sum_{i\in M}x_i^0\\ &=-(n-m)\sum_{i=1}^{n}x_i^0 \qquad + \quad (n+1)\sum_{i\in\{1,\dots,n\}\setminus M}x_i^0\end{aligned}$$

is an active hyperplane at $x_0\in\mathbb{C}\mathrm{P}$. □

4. The Normal Fan of the Configuration Polytope

For a closed convex subset $Q\subset\mathbb{R}^n$ let us denote by

$$p_Q:\mathbb{R}^n\longrightarrow Q \quad\text{with}\quad p_Q(x)=\{z\in Q\mid \inf_{q\in Q}\|x-q\|=\|z-x\|\}$$

the *metric projection* onto Q with respect to a norm $\|\cdot\|$ on $\mathbb{R}^n$. We will assume that $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$. Then the metric projection is a single valued mapping, because the Euclidean norm is strictly convex.

For a closed convex subset $Q\subset\mathbb{R}^n$ and $x\in Q$ we denote by

$$N(x)=-x+p_Q^{-1}(x)$$

the *normal cone* of Q at x and by

$$\Sigma(Q)=\{\tau \mid \tau \text{ is a face of } N(x),\ x\in Q\}$$

the *normal fan* of Q.

PROPOSITION 4.1 *The normal fan* $\Sigma(\mathbb{C}\mathrm{P})$ *of the configuration polytope* $\mathbb{C}\mathrm{P}=\mathbb{P}-\mathbb{P}$ *consists of the following cones* σ_x *together with its faces for the following points* $x\in\mathbb{C}\mathrm{P}$*:*

(i) *For* $x=e_i-e_j\in\mathbb{C}\mathrm{P}$
the cone is $\sigma_x=\mathrm{conv}\{a\in\mathcal{J} \mid l_j(a)<l_i(a)\}$.
(ii) *For* $x=\sum_{j=1}^n e_j+e_i\in\mathbb{C}\mathrm{P}$
the cone is $\sigma_x=\mathrm{conv}\{a\in\mathcal{J} \mid l_i(a)<l_{n+1}(a)\}$.
(iii) *For* $x=-\sum_{j=1}^n e_j-e_i\in\mathbb{C}\mathrm{P}$
the cone is $\sigma_x=\mathrm{conv}\{a\in\mathcal{J} \mid l_{n+1}(a)<l_i(a)\}$.

Here $l_i(x)=x_i$ *for* $i\in\{1,\ldots,n\}$ *and* $l_{n+1}(x)=-\sum_{i=1}^n x_i$ *with* $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$.

Proof. The extreme points of $\mathbb{C}\mathrm{P}$ are

$e_i-e_j\in\mathbb{C}\mathrm{P},\quad i,j\in\{1,\ldots,n\}$ and $i\neq j$,
$x=\left(\sum_{j=1}^n e_j\right)+e_i\in\mathbb{C}\mathrm{P},\quad i\in\{1,\ldots,n\}$,
$x=-\left(\sum_{j=1}^n e_j\right)-e_i\in\mathbb{C}\mathrm{P},\quad i\in\{1,\ldots,n\}$.

From Proposition 2.1 it follows that every extreme point of $\mathbb{C}\mathrm{P}$ is the intersection of $2^{(n-1)}$ distinct $(n-1)$-dimensional faces of $\mathbb{C}\mathrm{P}$. For the extreme point $e_i-e_j\in\mathbb{C}\mathrm{P}$ this $(n-1)$-dimensional faces are determined by the constraints

$$\langle a,e_i-e_j\rangle=n+1,\quad \text{with}\quad a\in\mathcal{J}.$$

Now by Proposition 2.1 the conditions $a\in\mathcal{J}$ and $\langle a,e_i-e_j\rangle=n+1$ are equivalent to $a\in\mathcal{J}$ and $l_j(a)<l_i(a)$. Since the normal cone of $\mathbb{C}\mathrm{P}$ at $e_i-e_j\in\mathbb{C}\mathrm{P}$ is the convex cone generated by the outer normal vectors of the adjacent $(n-1)$-dimensional faces at $e_i-e_j\in\mathbb{C}\mathrm{P}$ it follows that $N(e_i-e_j)=\mathrm{conv}\{a\in\mathcal{J} \mid l_j(a)<l_i(a)\}$ holds for the normal cone at $e_i-e_j\in\mathbb{C}\mathrm{P}$.

Points (ii) and (iii) can be proved in the same way, because for all subsets $M\subset\{1,\ldots,n\}$ with $1\leqslant m=\mathrm{card}\ M$ and $i\in M$

$$\langle a_M,z\rangle=n+1$$

holds with $z=\sum_{j=1}^{n} e_j + e_i \in \mathbb{C}\mathrm{P}$. □

Let Σ and Σ' be two fans in $\mathbb{R}^n$. Then Σ' is called a *refinement* of Σ if for every $\tau\in\Sigma'$ there exists a $\sigma\in\Sigma$ such that $\tau\subseteq\sigma$.

THEOREM 4.2 *The Morse fan Σ_n is a refinement of the normal fan $\Sigma(\mathbb{C}\mathrm{P})$.*

Proof. Every cone with nonempty interior of $\Sigma(\mathbb{C}\mathrm{P})$ is of the form

$$\sigma_{i,j} = \mathrm{conv}\{a\in\mathcal{J} \mid l_j(a) < l_i(a)\} \quad \text{for} \quad i,j\in\{1,\dots,n,n+1\} \quad \text{with} \quad i\neq j,$$

where $l_i(x)=x_i$ for $i\in\{1,\dots,n\}$ and $l_{n+1}(x)=-\sum_{i=1}^{n} x_i$ with $x=(x_1,\dots,x_n)\in\mathbb{R}^n$.

Now observe that every cone $\sigma_{i,j}$ is the union of $(n-1)!$ permutation cones

$$\sigma_\pi = \{x\in\mathbb{R}^n \mid l_{\pi(1)}(x) \leqslant l_{\pi(2)}(x) \leqslant \cdots \leqslant l_{\pi(n+1)}(x)\}$$

for

$$\pi\in\Pi_{i,j}(n+1) = \{\rho\in\Pi(n+1) \quad \text{with} \quad \rho(i)=i \quad \text{and} \quad \rho(j)=j\ \},$$

i.e.

$$\sigma_{i,j} = \bigcup_{\pi\in\Pi_{i,j}(n+1)} \sigma_\pi.$$

Hence Σ_n is a refinement of $\Sigma(\mathbb{C}\mathrm{P})$. □

REMARK 4.3 For dimension $n=2$ the Morse fan Σ_2 coincides with the fan $\Sigma(\mathbb{C}\mathrm{P})$ in $\mathbb{R}^2$. For dimension $n=3$ the polytope $\mathbb{C}\mathrm{P}\subset\mathbb{R}^3$ has 12 normal cones with nonempty interiors and each of this cones is the union of two permutation cones of the Morse fan Σ_3. In general the polytope $\mathbb{C}\mathrm{P}\subset\mathbb{R}^n$ has $n(n+1)$ normal cones with nonempty interiors and each of these cones is the union of $(n-1)!$ permutation cones of the Morse fan Σ_n.

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